MA114 Summer 2018
Worksheet 19 – Arc Length – 7/17/18

1. Find the exact length of each of the following curves.

   a) $z = y^{3/2}, 0 \le y \le 2$

$$S = \int_0^2 \sqrt{1+(z')^2}\,dy = \int_0^2 \sqrt{1+(\tfrac{3}{2}y^{1/2})^2}\,dy = \int_0^2 \sqrt{1+\tfrac{9}{4}y}\,dy$$

$$= \frac{4}{9}\cdot\frac{2}{3}\left(1+\tfrac{9}{4}y\right)^{\frac{3}{2}}\Big|_0^2$$

$$= \frac{8}{27}\left[\left(1+\tfrac{9}{2}\right)^{\frac{3}{2}} - 1\right]$$

   b) $z = \ln(1-x^2), 0 \le x \le \frac{1}{2}$

$$S = \int_0^{1/2} \sqrt{1+(z')^2}\,dx = \int_0^{1/2} \sqrt{1+\left(\frac{-2x}{1-x^2}\right)^2}\,dx = \int_0^{1/2} \sqrt{\frac{(1-x^2)^2+4x^2}{(1-x^2)^2}}\,dx$$

$$= \int_0^{1/2} \sqrt{\frac{(1-2x^2+x^4+4x^2)}{(1-x^2)^2}}\,dx$$

$$= \int_0^{1/2} \sqrt{\frac{(1+x^2)^2}{(1-x^2)^2}}\,dx$$

$$= \int_0^{1/2} \frac{1+x^2}{1-x^2}\,dx$$

$$= \int_0^{1/2} -1 + \frac{2}{1-x^2}\,dx$$

$$= -x\Big|_0^{1/2} + \int_0^{1/2} \frac{2}{(1-x)(1+x)}\,dx$$

$$= -\frac{1}{2} + \int_0^{1/2} \frac{1}{1-x} + \frac{1}{1+x}\,dx$$

$$= -\frac{1}{2} + \left[-\ln|1-x| + \ln|1+x|\Big|_0^{1/2}\right]$$

$$= -\frac{1}{2} - \ln\left(\tfrac{1}{2}\right) + \ln\left(\tfrac{3}{2}\right) + \ln(1) - \ln(1)$$

$$= -\frac{1}{2} + \ln\left(\frac{\frac{3}{2}}{\frac{1}{2}}\right) = \boxed{-\frac{1}{2} + \ln(3)}$$

   c) $w = 1 - e^{-t}, 0 \le t \le 2$ $\quad w' = e^{-t}$

$$S = \int_0^2 \sqrt{1+(w')^2}\,dt$$

$$= \int_0^2 \sqrt{1+e^{-2t}}\,dt \qquad u = 1+e^{-2t}$$

$$du = -2e^{-2t}\,dt$$

$$\frac{du}{-2(u-1)} = dt$$

$$= \int -\frac{1}{2}\frac{\sqrt{u}}{u-1}\,du$$

See last page.

d) $36y^2 = (x^2-4)^3, 2 \le x \le 3, y \ge 0$

$y = \frac{1}{6}(x^2-4)^{\frac{3}{2}}$

$y' = \frac{1}{4}(x^2-4)^{\frac{1}{2}} \cdot 2x$

$s = \int_2^3 \sqrt{1 + \left(\frac{x}{2}\sqrt{x^2-4}\right)^2}\, dx$

$= \int_2^3 \sqrt{1 + \frac{x^2(x^2-4)}{4}}\, dx$

$= \int_2^3 \sqrt{\frac{4+x^4-4x^2}{4}}\, dx$

$= \int_2^3 \sqrt{\frac{(x^2-2)^2}{2^2}}\, dx$

$= \int_2^3 \frac{x^2-2}{2}\, dx$

$= \frac{1}{2}\left(\frac{x^3}{3} - 2x\right)\Big|_2^3$

$= \frac{1}{2}\left(9 - 6 - \frac{8}{3} + 4\right)$

$= \boxed{\frac{13}{6}}$

e) $z = \ln(\cos(x)), 0 \le x \le \pi/3$

$\left(\frac{dz}{dx}\right) = \frac{-\sin(x)}{\cos(x)} = -\tan(x)$

$s = \int_0^{\pi/3} \sqrt{1 + \left(\frac{dz}{dx}\right)^2}\, dx$

$= \int_0^{\pi/3} \sqrt{1 + \tan^2(x)}\, dx$

$= \int_0^{\pi/3} \sec(x)\, dx$

$= \ln|\sec(x) + \tan(x)|\Big|_0^{\pi/3}$

$= \boxed{\ln|2 + \sqrt{3}|}$

f) $y = \frac{x^3}{3} + \frac{1}{4x}, 1 \le x \le 2$

~~$y' = x^2 - \frac{1}{4x^2}$~~

c) $s = \int_0^2 \sqrt{1+(\omega')^2}\, dt$

$= \int_0^2 \sqrt{1+e^{-2t}}\, dt$

$u = 1+e^{-2t}$ ~~t=0: u=...~~ $t=0: u=\sqrt{2}$

$du = -2e^{-2t}\,dt$ $t=2: u=\sqrt{1+e^{-4}}$

$\frac{du}{-2e^{-2t}} = dt$

$\frac{du}{-2(u-1)} = dt$

So $s = \int_{\sqrt{2}}^{\sqrt{1+e^{-4}}} \frac{\sqrt{u}}{-2(u-1)}\, du$

Let $r = \sqrt{u}$ $u = r^2$, so $u=\sqrt{2} \Rightarrow r=2$

$dr = \frac{1}{2\sqrt{u}}\, du$ $u=\sqrt{1+e^{-4}} = r = 1+e^{-4}$

$du = 2r\,dr$

Then $S = \int_{1+e^{-4}}^{2} \frac{r}{r^2-1} r\,dr$

$= \int_{1+e^{-4}}^{2} 1 + \frac{1}{r^2-1}\, dr$

$= \int_{1+e^{-4}}^{2} 1 + \frac{1}{2}\left(\frac{1}{r-1} - \frac{1}{r+1}\right) dr$

$= r + \frac{1}{2}\left(\ln|r-1| - \ln|r+1|\right)\Big|_{1+e^{-4}}^{2}$

$= 2 + \frac{1}{2}\left(\ln\left|\frac{2-1}{2+1}\right|\right) - (1+e^{-4}) - \frac{1}{2}\left(\ln\left(\frac{1+e^{-4}-1}{1+e^{-4}+1}\right)\right)$

$= \boxed{1 - e^{-4} + \ln\left(\sqrt{\tfrac{1}{3}}\right) - \ln\left(\left(\frac{2+e^{-4}}{e^{-4}}\right)^{1/2}\right)}$

$\frac{A}{r+1} + \frac{B}{r-1} = \frac{1}{r^2-1}$

$A(r-1) + B(r+1) = 1$

$r=-1$: $A = -\frac{1}{2}$

$r=1$: $B = +\frac{1}{2}$

f) $y' = x^2 - \frac{1}{4x^2}$

$s = \int_1^2 \sqrt{1+(y')^2}\, dx$

$= \int_1^2 \sqrt{1+\left(x^2 - \frac{1}{4x^2}\right)^2}\, dx$

$= \int_1^2 \sqrt{1 + x^4 - \frac{1}{2} + \frac{1}{16x^4}}\, dx$

$= \int_1^2 \sqrt{x^4 + \frac{1}{2} + \frac{1}{16}x^4}\, dx$

$= \int_1^2 \sqrt{\left(x^2 + \frac{1}{4x^2}\right)^2}\, dx$

$= \int_1^2 x^2 + \frac{1}{4x^2}\, dx$

$= \frac{x^3}{3} - \frac{1}{4x}\Big|_1^2$

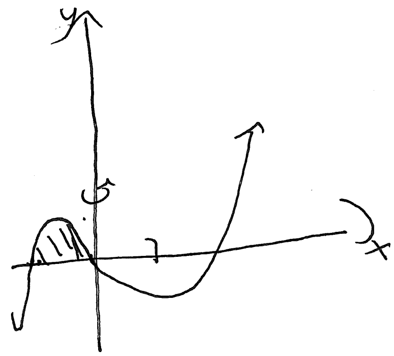
$= \frac{8}{3} - \frac{1}{8} - \frac{1}{3} + \frac{1}{4}$

$= \frac{7}{3} + \frac{1}{8}$

$= \frac{56+3}{24}$

$= \boxed{\frac{59}{24}}$

$y = x^3 - x^2 - 2x$
$= x(x^2 - x - 2)$
$= x(x-2)(x+1)$



$\int_0^1 2\pi x \sqrt{1 + \frac{1}{x^2}}$

$\frac{2\pi x\sqrt{1+x^2}}{x}$

$V = \int_{-1}^{0} 2\pi \cancel{x}(-x)(x^3 - x^2 - 2x)\, dx$

$= \int_{-1}^{0} 2\pi(-x^4 + x^3 + 2x^2)\, dx$

$= 2\pi\left(-\frac{x^5}{5} + \frac{x^4}{4} + \frac{2}{3}x^3\right)\Big|_{-1}^{0}$

$= -2\pi\left(\frac{1}{5} + \frac{1}{4} - \frac{2}{3}\right)$

$-2\pi\left(\frac{12 + 15 - 40}{60}\right)$ 27

$= \boxed{\frac{13\pi}{30}}$

$x = e^{-y}$
$\ln(x) = -y$
$y = -\ln(x)$
$\frac{dy}{dx} = -\frac{1}{x}$

$\frac{31}{40}$

dv u
$x \ln x$

$\frac{1}{2}x^2 \ln x - \int \frac{1}{2} x\, dx$